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## LOCALLY COMPACT GROUPS APPROXIMABLE BY SUBGROUPS ISOMORPHIC TO $\mathbb{Z}$ OR $\mathbb{R}$

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**ABSTRACT.** Let  $G$  be a locally compact topological group,  $G_0$  the connected component of its identity element, and  $\text{comp}(G)$  the union of all compact subgroups. A topological group will be called inductively monothetic if any subgroup generated (as a topological group) by finitely many elements is generated (as a topological group) by a single element. The space  $\text{SUB}(G)$  of all closed subgroups of  $G$  carries a compact Hausdorff topology called the Chabauty topology. Let  $\mathcal{F}_1(G)$ , respectively,  $\mathcal{R}_1(G)$ , denote the subspace of all discrete subgroups isomorphic to  $\mathbb{Z}$ , respectively, all subgroups isomorphic to  $\mathbb{R}$ . It is shown that a necessary and sufficient condition for  $G \in \overline{\mathcal{F}_1(G)}$  to hold is that  $G$  is abelian, and either that  $G \cong \mathbb{R} \times \text{comp}(G)$  and  $G/G_0$  is inductively monothetic, or else that  $G$  is discrete and isomorphic to a subgroup of  $\mathbb{Q}$ . It is further shown that a necessary and sufficient condition for  $G \in \overline{\mathcal{R}_1(G)}$  to hold is that  $G \cong \mathbb{R} \times C$  for a compact connected abelian group  $C$ .

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## 1. PREFACE

The simplest group arising directly from the activity of counting is the group  $\mathbb{Z}$  of integers. On the other hand, one of the more sophisticated concepts of group theory is that of a locally compact topological group; it evolved widely and deeply since David Hilbert in 1900 posed the question whether a locally euclidean topological group might be parametrized differentiably so that the group operations become differentiable. Bringing  $\mathbb{Z}$  and locally compact groups together in a topologically systematic fashion is made possible by a compact Hausdorff space  $\mathbf{SUB}(G)$  attached to a locally compact group  $G$  in a very natural fashion, namely, as the set of closed subgroups endowed with a suitably defined topology. Since  $G$  itself is a prominent element of  $\mathbf{SUB}(G)$ , we pose and answer completely the question under which circumstances  $G$  can be approximated in  $\mathbf{SUB}(G)$  by those subgroups of  $G$  which are isomorphic to  $\mathbb{Z}$ . A topological parameter attached to a topological space  $X$  is the smallest cardinal of a basis for the collection of all open subsets; this parameter is called the weight  $w(X)$  of  $X$ . If the weight of  $X$  is not bigger than the first infinite cardinal, one says that  $X$  satisfies the Second Axiom of Countability. Our findings about the approximability of  $G$  by subgroups isomorphic to  $\mathbb{Z}$  will show a fact that one would not expect at first glance: Such an approximability does not impose any bound whatsoever on the weight  $w(G)$  of  $G$ .

If one ascends from the group  $\mathbb{Z}$  of counting numbers to the group  $\mathbb{R}$  of real numbers which permits us to measure lengths and distances, then the completely analogous question suggests itself, asking indeed which locally compact groups  $G$  can be approximated in  $\mathbf{SUB}(G)$  by subgroups isomorphic to  $\mathbb{R}$ . Again we answer that question completely and find, that the answer to the second question yields a simpler structure than the answer to the first question.

The answers are described in the Abstract which precedes our text. It is relatively immediate that our discussion will usher us into the domain of abelian locally compact groups. But the final proofs do lead us more deeply into the structure of these groups than one might anticipate and are, therefore, also longer than expected. With these remarks, we now turn to the details.

## 2. PRELIMINARIES

**2.1. Basic concepts and definitions.** Let  $G$  be a locally compact group and  $\text{comp}(G)$  the union of its compact subgroups. By Weil's Lemma ([14, Proposition 7.43]), an element  $g \in G$  is either contained

in  $\text{comp}(G)$  or else the group  $\langle g \rangle$  is isomorphic as a topological group to  $\mathbb{Z}$ . Subgroups of this kind we shall call *integral*. A subgroup  $E$  of  $G$  is called *real* if it is isomorphic to  $\mathbb{R}$  as a topological group.

We denote by  $\mathbf{SUB}(G)$  the space of closed subgroups of  $G$  equipped with the *Chabauty topology*; this is a compact space. In this space, each closed subgroup  $H$  of  $G$  has a neighborhood base consisting of sets

$$\mathcal{U}(H; K, W) \stackrel{\text{def}}{=} \{L \in \mathbf{SUB}(G) \mid L \cap K \subseteq WH \text{ and } H \cap K \subseteq WL\}, \quad (2.1)$$

where  $K$  ranges through the set  $\mathcal{K}$  of all compact subsets of  $G$  and  $W$  through the set  $\mathcal{U}(e)$  of all neighborhoods of the identity. In particular  $G \in \mathbf{SUB}(G)$  and the singleton subgroup  $E = \{1\}$  have bases for their respective neighborhoods of the form

$$\mathcal{U}(G; K, W) = \{L \in \mathbf{SUB}(G) \mid K \subseteq WL\}, \quad (2.2)$$

$$\mathcal{U}(E; K, W) = \{L \in \mathbf{SUB}(G) \mid L \cap K \subseteq W\}, \quad (2.3)$$

$K \in \mathcal{K}$  and  $W \in \mathcal{U}(e)$ .

*Remark 2.1.* Returning for a moment to Equation (2.1) we assume that  $G$  is  $\sigma$ -compact and satisfies the First Axiom of Countability. Then  $G$  contains a sequence  $(K_m)_{m \in \mathbb{N}}$  of compact subsets whose interiors form an ascending sequence of open sets covering  $G$ , and there is a sequence  $(W_n)_{n \in \mathbb{N}}$  of open identity neighborhoods forming a basis of the filter of identity neighborhoods. Then each element  $H \in \mathbf{SUB}(G)$  has a countable basis  $(\mathcal{U}(H; K_m, W_n))_{m, n \in \mathbb{N}}$  for its neighborhood filter according to Equation (2.1). Thus  $\mathbf{SUB}(G)$  satisfies the First Axiom of Countability.

We denote by  $\mathcal{F}_1(G)$  the subspace of  $\mathbf{SUB}(G)$  containing all  $\langle g \rangle$  with  $g \in G \setminus \text{comp}(G)$ , that is, all subgroups isomorphic to the discrete group  $\mathbb{Z}$  of all integers, the *free group of rank 1*.

**Example 2.2** (The additive group  $\mathbb{R}$ ). The mapping  $\phi_{\mathbb{R}}: [0, \infty] \rightarrow \mathbf{SUB}(\mathbb{R})$  defined by

$$\phi_{\mathbb{R}}(r) = \begin{cases} \frac{1}{r} \cdot \mathbb{Z} & \text{if } 0 < r < \infty, \\ \{0\} & \text{if } r = 0, \\ \mathbb{R} & \text{if } r = \infty \end{cases}$$

is a homeomorphism (see Proposition 1.7 of [11]). Here  $\text{comp}(G) = \{0\}$ , and  $\mathcal{F}_1(\mathbb{R}) = \{\langle r \rangle \mid 0 < r\}$ , and if  $(r_n)_{n \in \mathbb{N}}$  is a sequence of real numbers converging to 0, then  $(\langle r_n \rangle)_{n \in \mathbb{N}}$  converges to  $\mathbb{R}$ . Therefore,  $\mathbb{R} \in \overline{\mathcal{F}_1(\mathbb{R})}$ .

**Definition 2.3.** A locally compact group  $G$  is said to be *integrally approximable* if  $G \in \overline{\mathcal{F}_1(G)}$ .

Here is an equivalent way of expressing that  $G$  is integrally approximable:

*There is a net  $(S_j)_{j \in J}$  of subgroups isomorphic to  $\mathbb{Z}$  in  $G$  such that  $G = \lim_{j \in J} S_j$  in  $\mathbf{SUB}(G)$ .*

*Remark 2.4.* If  $G$  is integrally approximable,  $\overline{\mathcal{F}_1(G)} \neq \emptyset$ , and so  $G \neq \text{comp}(G)$ . In particular,  $G$  is not singleton.

Our objective is to describe precisely which locally compact groups are integrally approximable. The outcome is anticipated in the abstract. It may be instructive for our intuition to consider some examples right now.

**2.2. Some examples.** We noticed in Example 2.2 above that  $\mathbb{R}$  is integrally approximable. A bit more generally, we record

**Example 2.5.** For  $n \in \mathbb{N}$  and  $G = \mathbb{R}^n$ , the following statements are equivalent:

- (1)  $G$  is integrally approximable.
- (2)  $n = 1$ .

*Proof.* By Example 2.2, (2) implies (1). For proving the reverse implication, we define the elements  $e_k = (\delta_{km})_{m=1, \dots, n}$ ,  $k = 1, \dots, n$  for the Kronecker deltas

$$\delta_{km} = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise.} \end{cases}$$

Now we assume (1) and  $n \geq 2$  and propose to derive a contradiction. We let  $W \in \mathcal{U}((0, \dots, 0))$  be the open ball of radius  $\frac{1}{2}$  with respect to the euclidean metric on  $\mathbb{R}^n$ , and let  $K$  be the closed ball of radius 2 around  $(0, \dots, 0)$ . By (1) there is an integral subgroup  $S$  in the neighborhood  $\mathcal{U}(G; K, W)$  of  $G$  in  $\mathbf{SUB}(G)$ .

In view of Equation (2.2) this means  $K \subset W + S$ . Since  $n \geq 2$  we know that  $e_1$  and  $e_2$  are contained in  $K$  and therefore in  $W + S$ , that is there are elements  $w_1, w_2 \in W$  such that  $s_1 = e_1 - w_1$  and  $s_2 = e_2 - w_2$  are contained  $S$ . Now the euclidean distance of  $s_m$  from  $e_m$  for  $m = 1, 2$  is  $< \frac{1}{2}$  in the euclidean plane  $E_2 \cong \mathbb{R}^2$  spanned by  $e_1$  and  $e_2$ . Therefore, the two elements  $s_1$  and  $s_2$  are linearly independent. On the other hand, being elements of the subgroup  $S \cong \mathbb{Z}$ , they must be linearly dependent. This contradiction proves that (1) implies (2).  $\square$

**Example 2.6.** The group  $\mathbb{Q}$  (with the discrete topology) is integrally approximable.

*Proof.* For each natural number  $n$  set  $H_n = \frac{1}{n!}\mathbb{Z}$ . Since  $(H_n)$  is an increasing sequence and

$$\bigcup_{n \in \mathbb{N}} H_n = \mathbb{Q},$$

we also have  $\lim_{n \in \mathbb{N}} H_n = \mathbb{Q}$  as we may conclude directly using (2.2) or by invoking Proposition 2.10 of [10]. This proves the claim.  $\square$

**Example 2.7.** The group  $G = \mathbb{R} \times \mathbb{Z}(p)^n$  for any prime  $p$  and any natural number  $n \geq 2$  is not approximable by integral subgroups. The smallest example in this class is  $\mathbb{R} \times \mathbb{Z}(2)^2$ .

*Proof.* We note that  $\mathbb{Z}(p)^n$  is a vector space  $V$  of dimension  $n$  over the field  $\mathbb{F} = \text{GF}(p)$ . By way of contradiction suppose that  $G$  is integrally approximable. Let  $W = [-1, 1] \times \{0\}$  and let  $K = \{0\} \times B$  for a basis  $B$  of  $V$ . Since  $G$  is integrally approximable, there is a  $Z \in \mathcal{F}_1(G)$  such that  $Z \in \mathcal{U}(G; K, W)$ , that is,  $K \subseteq W + Z$ . There is an  $r \in \mathbb{R}$  and a  $v \in V$  such that  $Z = \mathbb{Z} \cdot (r, v)$ . Thus

$$\begin{aligned} \{0\} \times B &\subseteq ([-1, 1] \times \{0\}) + \mathbb{Z} \cdot (r, v) \\ &= \bigcup_{n \in \mathbb{Z}} \{nr + [-1, 1]\} \times \{n \cdot v\} \subseteq \mathbb{R} \times \mathbb{F} \cdot v, \end{aligned}$$

and thus  $B \in \mathbb{F} \cdot v$ . This implies  $\dim_{\mathbb{F}} V \leq 1$  in contradiction to the assumption  $n \geq 2$ .  $\square$

For any prime  $p$  we recall the basic groups  $\mathbb{Z}(p^n)$ ,  $n = 1, \dots, \infty$ , and  $\mathbb{Z}_p \subset \mathbb{Q}_p$ , where  $\mathbb{Z}(p^\infty)$  is the divisible Prüfer group and  $\mathbb{Z}_p$  its character group, the group of  $p$ -adic integers (see. e.g. [14], Example 1.38(i), p. 27) and where  $\mathbb{Q}_p$  denotes the group of  $p$ -adic rationals (see loc. cit. Exercise E1.16, p. 27). Then Example 2.7 raises at once the following question:

**Question 2.8.** *Are the locally compact groups  $G = \mathbb{R} \times C$  for  $C = \mathbb{Z}(p^n)$ ,  $C = \mathbb{Z}(p^\infty)$ ,  $C = \mathbb{Z}_p$ , or  $C = \mathbb{Q}_p$  integrally approximable?*

In the light of the negative Example 2.7, this may not appear so simple a matter to answer. Our results will show that they all are integrally approximable.

While the group  $\mathbb{Z}$  is integrally approximable trivially from the definition, we have, in the context of the group  $\mathbb{Z}$ , the following lemma, whose proof we can handle as an exercise directly from the definitions and which serves as a further example of the particular role played by  $\mathbb{Z}$  in the context of integrally approximable groups.

**Lemma 2.9.** *Let  $A$  be a locally compact abelian group and assume that  $A \times \mathbb{Z}$  is integrally approximable. Then  $|A| = 1$ .*

*Proof.* By way of contradiction assume that there is an  $a \neq 0$  in  $A$ . Then there is a zero-neighborhood  $W = -W$  in  $A$  such that  $a \notin W + W$ . Set  $G = A \times \mathbb{Z}$  and  $K = \{(a, 1), (0, 1)\} \subseteq G$ . Since  $G$  is integrally approximable there is a  $Z \cong \mathbb{Z}$  in  $\mathcal{F}_1(G)$  such that

$$Z \in \mathcal{U}(G; K, W \times \{0\}).$$

Since  $Z$  is an integral subgroup, there are elements  $b \in A$  and  $0 < n \in \mathbb{Z}$  such that  $Z = \mathbb{Z} \cdot (b, n)$ . So  $(a, 1)$  and  $(0, 1)$  are contained in  $(W \times \{0\}) + \mathbb{Z} \cdot (b, n)$ . Therefore there are elements  $w_a, w_0 \in W$  and integers  $m_a, m_0 \in \mathbb{Z}$  such that

- (i)  $w_0 + m_0 \cdot b = 0$ ,
- (ii)  $w_a + m_a \cdot b = a$ , and
- (iii)  $m_0 n = 1$ .
- (iv)  $m_a n = 1$ .

Equations (iii) and (iv), holding in  $\mathbb{Z}$ , imply  $m_0 = m_a = n = 1$ . Thus equation (i) implies  $b \in -W = W$ . Then from (ii) it follows that  $a \in W + W$ . This is a contradiction, which proves our claim for the example.  $\square$

### 3. THE CLASS OF APPROXIMABLE GROUPS

As a first step towards the main results it will be helpful to observe the available closure properties of the class of integrally approximable groups.

#### 3.1. Preservation properties.

**Proposition 3.1.** *The class of integrally approximable locally compact groups is closed under the following operations:*

- (OS) *Passing to open nonsingleton subgroups,*
- (QG) *passing to quotients modulo compact subgroups,*
- (QO) *passing to torsion-free quotients modulo open subgroups,*
- (DU) *forming directed unions of closed subgroups,*
- (PL) *forming strict projective limits of quotients  $G/N$  modulo compact subgroup  $N$ .*

*Proof.* (OS) Let  $G$  be a locally compact group  $G$  that is approximable by integral subgroups and let  $U$  be an open nonsingleton subgroup. Let  $K$  be a compact subspace of  $U$  and  $W$  an identity neighborhood

contained in  $U$ ; we may take  $K \neq \{1\}$  and  $W$  small enough so that  $K \not\subseteq W$ . We must find a subgroup  $Z \cong \mathbb{Z}$  inside  $U$  such that  $Z \in \mathcal{U}(U; K, W)$ , that is,  $K \subseteq WZ$ .

However,  $G$  is integrally approximable, and so there is a subgroup  $E \cong \mathbb{Z}$  of  $G$  such that  $E \in \mathcal{U}(G; K, W)$ , that is,  $K \subseteq WE$ . Then  $K = K \cap U \subseteq WE \cap U$ . By the modular law,  $W(E \cap U) = WE \cap U$  and so  $K \subseteq W(E \cap U)$ . The condition  $K \not\subseteq W$  rules out the possibility that  $E \cap U = \{1\}$ . Since  $E \cong \mathbb{Z}$  we know that  $E \cap U \cong \mathbb{Z}$  and so we may take  $Z = E \cap U$  and thus obtain  $K \subseteq WZ$  which is what we have to prove.

(QG) Let  $N$  be a compact subgroup of a locally compact group  $G$  approximable by integral subgroups and let  $\pi: G \rightarrow H$ ,  $H = G/N$ , be the quotient morphism. Then the continuity of the map  $A \mapsto \pi(A) : \mathcal{SUB}(G) \rightarrow \mathcal{SUB}(H)$  (see Corollary 2.4 of [7]) implies that  $H$  is approximable by integral subgroups.

(QO) Let  $U$  be an open subgroup of a locally compact group  $G$  approximable by integral subgroups so that  $H \stackrel{\text{def}}{=} G/U$  is torsion-free, and let  $\pi: G \rightarrow H$  be the quotient morphism. Since  $U$  is open,  $H$  is discrete, and the singleton set containing the identity  $\tilde{e} = U$  in  $H = G/U$  is an identity neighborhood. So for a given compact, hence finite, subset  $\tilde{K}$  of  $H$  we have to find a  $\tilde{Z} \in \mathcal{F}_1(H)$  with  $\tilde{Z} \subseteq \mathcal{U}(H; \tilde{K}, \{\tilde{e}\})$ , that is  $\tilde{K} \subseteq \tilde{Z}$  according to Equation (2.2). We may and will assume that there is at least one  $\tilde{k} \in \tilde{K}$  such that  $\tilde{k} \neq \tilde{e}$ .

Now by the local compactness of  $G$  we find a compact set  $K$  of  $G$  such that  $\pi(K) = \tilde{K}$ .

Since  $U$  is an identity neighborhood in  $G$  and since  $G$  is integrally approximable, there is a subgroup  $Z \in \mathcal{F}_1(H)$  contained in  $\mathcal{U}(G; K, U)$ , that is  $K \subseteq UZ$ . Applying  $\pi$ , we get  $\tilde{K} \subseteq \pi(Z)$ . In particular,  $\tilde{e} \neq \tilde{k} \in \pi(Z)$ . Since  $H$  is torsion-free and  $Z \cong \mathbb{Z}$  we conclude that  $\pi(Z) \cong \mathbb{Z}$ , and so we can set  $\tilde{Z} = \pi(Z)$ , getting  $\tilde{K} \subseteq \tilde{Z}$ , which we had to show.

(DU) Let  $(G_i)_{i \in I}$  be a directed family of closed subgroups of a locally compact group  $G$  such that  $G = \overline{\bigcup_{i \in I} G_i}$ . Then from Proposition 2.10 of [10] we know

$$G = \lim_i G_i \text{ in } \mathcal{SUB}(G). \quad (3.1)$$

Now assume

$$(\forall i \in I) \ G_i \text{ is approximable by integral subgroups.} \quad (3.2)$$



Let  $\mathcal{U}$  be an open neighborhood of  $G$  in  $\mathbf{SUB}(G)$ . Then by (3.1) there is some  $j \in I$  with  $G_j \in \mathcal{U}$ , and so  $\mathcal{U}$  is also an open neighborhood of  $G_j$  in  $\mathbf{SUB}(G)$ . Then by (3.2) there is a closed subgroup  $Z \cong \mathbb{Z}$  in  $G$  with  $Z \in \mathcal{U}$ . This proves that  $G$  is approximable by integral subgroups.

(PL) Let the locally compact group  $G$  be a strict projective limit  $G = \lim_{N \in \mathcal{N}} G/N$  of integrally approximable quotient groups modulo compact normal subgroups  $N$ . Then  $G$  has arbitrarily small open identity neighborhoods  $W$  for which there is an  $N \in \mathcal{N}$  such that  $W = NW$ . Let  $K \in \mathcal{K}$  be a compact subspace of  $G$ . If  $W$  is given, we note that  $NK$  is still compact, and so we assume that  $NK = K$  as well. We aim to show that there is a subgroup  $Z \cong \mathbb{Z}$  of  $G$  such that  $K \subseteq WZ$  which will show that  $Z \in \mathcal{U}(G; K, W)$  as in Equation (2.2), and this will complete the proof.

Now we assume that for all  $M \in \mathcal{N}$  the group  $G/M$  is approximable by integral subgroups. Then, in particular,  $G/N$  is approximable by integral subgroups. Therefore we find a subgroup  $Z_N \subseteq G/N$  such that  $Z_N \cong \mathbb{Z}$  and  $Z_N \in \mathcal{U}(G/N; K/N, W/N)$  according to (2.2). This means that

$$K/N \subseteq (W/N) \cdot Z_N. \quad (3.3)$$

Let  $z \in G$  be such that  $Z_N = \overline{\langle zN \rangle}$ . Set  $Z = \overline{\langle z \rangle}$ , then  $Z \cong \mathbb{Z}$  and  $Z_N = ZN/N$ . Then (3.3) is equivalent to  $K/N \subseteq (W/N)(ZN/N) = WZ/N$  which in turn is equivalent to  $K \subseteq WZ$  and this is what we had to show.  $\square$

*Remark 3.2.* In the proof of (QO) it is noteworthy that we did not invoke an argument claiming the continuity of the function  $A \mapsto AU/U : \mathbf{SUB}(G) \rightarrow \mathbf{SUB}(H)$ . Indeed let  $\pi : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{Z}$  be the projection. The sequence  $(\langle (n, 1) \rangle)_{n \in \mathbb{N}}$  converges to the trivial subgroup  $\{(0, 0)\}$  in  $\mathbf{SUB}(G)$ , but its image is the constant sequence with value  $\langle 1 \rangle = \mathbb{Z}$  and therefore converges to  $\mathbb{Z}$ . This shows that the induced map  $\mathbf{SUB}(\pi) : \mathbf{SUB}(G) \rightarrow \mathbf{SUB}(H)$  need not be continuous in general.

#### 4. THE CASE OF DISCRETE GROUPS

In order to demonstrate the workings of some of the operations discussed in Proposition 3.1 we show

**Lemma 4.1.** *Assume that an integrally approximable locally compact group  $G$  has a compact identity component  $G_0$ . Then  $G$  is discrete.*

*Proof.* By way of contradiction suppose that  $G$  is not discrete. Since  $G_0$  is compact, there is a compact open subgroup  $U$  (see [16], Lemma 2.3.1 on p. 54). Since  $G$  is not discrete,  $U \neq \{1\}$ . Since  $G$  is integrally



approximable, so is  $U$  by Proposition 3.1 (OS). Then  $U \neq \text{comp}(U)$  by Remark 2.4, but  $U$  being compact we have  $U = \text{comp}(U)$  and this is a contradiction which proves the lemma.  $\square$

**4.1. Monothetic and inductively monothetic groups.** Before we proceed we need to recall some facts around monothetic groups. A topological group  $G$  is *monothetic* if there is an element  $g \in G$  such that  $G = \overline{\langle g \rangle}$ .

**Definition 4.2** (Inductively monothetic group). A topological group  $G$  is called *inductively monothetic* if (and only if) every finite subset  $F \subseteq G$  there is an element  $g \in G$  such that  $\overline{\langle F \rangle} = \overline{\langle g \rangle}$ .

The circle group  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  contains a unique element  $t = 2^{-1} + \mathbb{Z} \in \mathbb{T}$  such that  $2 \cdot t = 0$ . The group  $\mathbb{T}^2$  is monothetic but not inductively monothetic, since the subgroup  $\{(t, 0), (0, t)\}$  is finitely generated but not monothetic. The discrete additive group  $\mathbb{Q}$  is inductively monothetic but is not monothetic.

In [12], Theorem 4.12 characterizes inductively monothetic locally compact groups. Before we cite this result we recall that Braconnier (see [1]) called a locally compact group  $G$  a *local product*  $\prod_{j \in J}^{\text{loc}} (G_j, C_j)$  for a family  $(G_j)_{j \in J}$  of locally compact groups  $G_j$  if each of them has a compact open subgroup  $C_j$  such that an element  $g = (g_j)_{j \in J} \in \prod_{j \in J} G_j$  is in  $G$  iff there is a finite subset  $F_g \subseteq J$  such that  $g_j \in C_j$  whenever  $j \notin F_g$ .

**Proposition 4.3** (Classification of inductively monothetic groups). *A locally compact group  $G$  is inductively monothetic if one of the following conditions is satisfied:*

- (1)  $G$  is a one-dimensional compact connected group,
- (2)  $G$  is discrete and is isomorphic to a subgroup of  $\mathbb{Q}$ .
- (3)  $G$  is isomorphic to a local product  $\prod_p^{\text{loc}} (G_p, C_p)$  where each of its characteristic  $p$ -primary components  $G_p$  is either  $\cong \mathbb{Z}(p^n)$ ,  $n = 0, 1, \dots, \infty$ , or  $\mathbb{Z}_p$ , or  $\mathbb{Q}_p$ .

It follows, in particular, that a *totally disconnected compact monothetic group is inductively monothetic* so that the concept of an inductively monothetic locally compact group is more general than that of a locally compact monothetic group *in the totally disconnected domain*. We remark that a locally compact group is called *periodic* if it is totally disconnected and has no subgroups isomorphic to  $\mathbb{Z}$ . Thus the class (3) of Proposition 4.3 covers precisely the periodic inductively monothetic groups.

Our present stage of information allows us to clarify on an elementary level the discrete side of our project:

**Theorem 4.4.** *Let  $G$  be a locally compact group such that  $G_0$  is a compact group. Then the following assertions are equivalent:*

- (1)  $G$  is integrally approximable.
- (2)  $G$  is discrete and isomorphic to a nonsingleton subgroup of  $\mathbb{Q}$ .

*Proof.* In Example 2.6 we saw that  $\mathbb{Q}$  is integrally approximable. Then from Proposition 3.1 (OS) it follows that every nonsingleton subgroup of  $\mathbb{Q}$  is integrally approximable. Thus (2)  $\Rightarrow$  (1).

We have to show (1) $\Rightarrow$ (2): Thus we assume (1). In particular,  $G$  is nonsingleton. Since the subgroup  $G_0$  is compact, Lemma 4.1 applies and shows that  $G$  is discrete. Then by Equation 2.2 in  $\mathbf{SUB}(G)$  the element  $G$  has a basis of neighborhoods  $\mathcal{U}(G; F, \{0\}) = \{H \in \mathbf{SUB}(G) : F \subseteq H\}$  as  $H$  ranges through the finite subsets of  $G$ . Since  $G$  is integrally approximable, there exists a  $Z \in \mathcal{F}_1(G)$  such that  $Z \in \mathcal{U}(G; F, \{0\})$ , that is,  $F \subseteq Z$ . Then  $\langle F \rangle$  is infinite cyclic as a subgroup of a group  $\cong \mathbb{Z}$ . Therefore  $G$  is discrete, torsion-free, and inductively monothetic. Then Proposition 4.3 shows that  $G$  is isomorphic to a subgroup of  $\mathbb{Q}$ .  $\square$

## 5. NECESSARY CONDITIONS

For the remainder of the effort to classify integrally approximable groups we may therefore concentrate on nondiscrete groups, and indeed on locally compact groups  $G$  whose identity component  $G_0$  is noncompact.

**5.1. Background on abelian locally compact groups.** We first point out why we have to focus on commutative locally compact groups. Indeed in [3, Proposition 3.4] the following fact was established:

**Proposition 5.1.** *Let  $G$  be a locally compact group. Then the space  $\mathbf{SUB}_{\text{ab}}(G)$  of closed abelian subgroups of  $G$  is closed in  $\mathbf{SUB}(G)$ .*

We have  $\mathcal{F}_1(G) \subseteq \mathbf{SUB}_{\text{ab}}(G)$  and thus  $\overline{\mathcal{F}_1(G)} \subseteq \mathbf{SUB}_{\text{ab}}(G)$ . Accordingly, in view of Definition 2.3 we therefore have

**Corollary 5.2.** *An integrally approximable locally compact group is abelian.*

Thus we now focus on locally compact abelian groups and their duality theory. As a consequence we shall henceforth write the groups we

discuss in additive notation. An example is the following result of Cornulier's (see [5], Theorem 1.1):

**Proposition 5.3** (Pontryagin-Chabauty Duality). *Let  $G$  be an abelian locally compact group. Then the annihilator map*

$$H \mapsto H^\perp : \mathcal{SUB}(G) \rightarrow \mathcal{SUB}(\widehat{G})$$

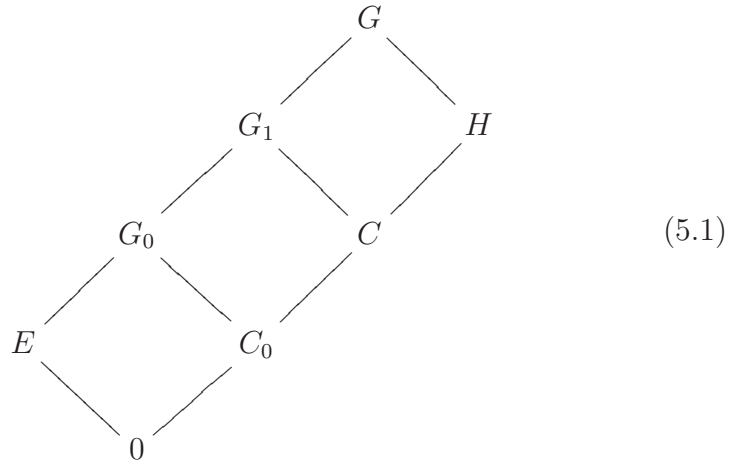
*is a homeomorphism.*

This will allow us to apply the so-called *annihilator mechanism* as discussed e.g. in [14], 7.12 ff., pp.314 ff..

What will be relevant in our present context is a main structure theorem for abelian groups (see [14], Theorem 7.57, pp. 345 ff.).

**Proposition 5.4.** *Every locally compact abelian group  $G$  is algebraically and topologically of the form  $G = E \oplus H$  for a subgroup  $E \cong \mathbb{R}^n$  and a locally compact abelian subgroup  $H$  which has the following properties*

- (a)  $H$  contains a compact subgroup which is open in  $H$ .
- (b)  $H$  contains  $\text{comp}(G)$ .
- (c)  $H_0 = (\text{comp}(G))_0 = \text{comp}(G_0)$  is the unique maximal compact connected subgroup of  $G$ .
- (d) The subgroup  $G_1 \stackrel{\text{def}}{=} G_0 + \text{comp}(G)$  is an open, hence closed, fully characteristic subgroup which is isomorphic to  $\mathbb{R}^n \times \text{comp}(G)$ .
- (e)  $G/G_1$  is a discrete torsion-free group and  $G_1$  is the smallest open subgroup with this property.



$$C = \text{comp}(G), \quad G_1 = G_0 + C = E \oplus C.$$

**5.2. Necessity.** These results allow us to narrow our scope onto integrally approximable groups  $G$  further and to derive necessary conditions for  $G$  to be integrally approximable.

**Proposition 5.5.** *Every nondiscrete integrally approximable locally compact abelian group  $G$  is algebraically and topologically of the form  $G = E \oplus \text{comp}(G)$  for a subgroup  $E \cong \mathbb{R}$  and the locally compact abelian subgroup  $\text{comp}(G)$ .*

*Proof.* (i) Using the notation of Proposition 5.4 (d) and (e) above we first claim  $G = G_1$ . By (d) the subgroup  $G_1 = G_0 + \text{comp}(G)$  is open, and by (e) the factor group  $G/G_1$  is torsion free. Suppose our claim is false. Then we find an element  $g \in G \setminus G_1$ . Then  $Z \stackrel{\text{def}}{=} \langle g \rangle$  is cyclic and  $(Z + G_1)/G_1 \cong Z/(Z \cap G_1)$  is torsion-free by (e), and so  $Z \cap G_1 = \{0\}$ . Then it follows from (d), since  $G_1$  is open, that  $Z$  is discrete. Hence  $U = G_1 + Z$  is a nonsingleton open subgroup  $\cong G_1 \times \mathbb{Z}$  by (e). Then by Lemma 2.9 we have  $G_1 = \{0\}$  which forces  $G$  to be discrete, contrary to our hypothesis.

(ii) Now by Proposition 5.4 again, we may identify  $G$  with  $\mathbb{R}^n \times H$  where  $H_0$  is compact and  $H = \text{comp}(H)$ . Now  $H$  contains a compact open subgroup  $U$  (cf. the proof of Lemma 4.1) and  $\mathbb{R}^n \times U$  is an open subgroup of  $G$  which is nonsingleton since  $G$  is nondiscrete. Hence  $\mathbb{R}^n \times U$  is integrally approximable by Proposition 3.1. Then the projection  $\mathbb{R}^n \times U \rightarrow \mathbb{R}^n$  is covered by part (QG) of Proposition 3.1 and thus we know that  $\mathbb{R}^n$  is integrally approximable. Then Example 2.5 shows that  $n = 1$ .  $\square$

*Remark 5.6.* If the relation  $G = G_0 + \text{comp}(G)$  holds for a locally compact abelian group  $G$ , then  $G/G_0 \cong \text{comp}(G)/\text{comp}(G)_0$ . Moreover,  $\text{comp}(G)$  is the union of compact subgroups being open in  $\text{comp}(G)$ .

From here on we concentrate on groups of the form  $\mathbb{R} \times H$  where  $H$  is an abelian locally compact group with  $H = \text{comp}(H)$ .

**Definition 5.7.** We call a locally compact group  $H$  *periodic* if it is totally disconnected and satisfies  $H = \text{comp}(H)$ .

Periodic abelian groups have known structure due to Braconnier (see [1]; cf. also [12]). Indeed, a periodic group  $G$  is (isomorphic to) a local product

$$\prod_{p \text{ prime}}^{\text{loc}} (G_p, C_p)$$

for the  $p$ -primary components (or  $p$ -Sylow subgroups)  $G_p$ .

**Lemma 5.8.** *Let  $G$  be an integrally approximable locally compact group such that  $\text{comp}(G)$  is periodic and compact. Then  $\text{comp}(G)$  is monothetic.*

*Proof.* Write  $H = \text{comp}(G)$ . Then we may identify  $H$  with the product  $\prod_p H_p$  of its compact  $p$ -primary components (see also [14], Proposition 8.8(ii)). A compact group  $H$  is monothetic iff there is a morphism  $\mathbb{Z} \rightarrow H$  with dense image iff (dually) there is an injective morphism  $\widehat{H} \rightarrow \mathbb{T}$ . As  $H$  is totally disconnected,  $\widehat{H}$  is a torsion group (see [14], Corollary 8.5, p. 377), and so the group  $\widehat{H}$  is embeddable into  $\mathbb{T}$  iff it is embeddable into the torsion group  $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}(p^\infty)$  of  $\mathbb{T}$  (see [14], Corollary A1.43(ii) on p. 694) iff each  $\widehat{H}_p$  is embeddable into  $\mathbb{Z}(p^\infty)$ . Hence  $H$  is monothetic iff each  $H_p$  has  $p$ -rank  $\leq 1$ . By way of contradiction suppose that this is not the case and that there is a prime  $p$  such that the  $p$ -rank of  $H_p$  is  $\geq 2$ . So the compact group  $H/p \cdot H$  has exponent  $p$  and is isomorphic to a power  $\mathbb{Z}(p)^I$  with  $\text{card } I \geq 2$ . Therefore we have a projection of  $H/p \cdot H \cong \mathbb{Z}(p)^I$  onto  $\mathbb{Z}(p)^2$ . This provides a surjective morphism  $H \rightarrow H/p \cdot H \cong \mathbb{Z}(p)^I \rightarrow \mathbb{Z}(p)^2$ . Thus Proposition 5.5 tells us that  $G = \mathbb{R} \times H$  has a quotient group  $\mathbb{R} \times \mathbb{Z}(p)^2$  modulo a compact kernel. By Proposition 3.1 (QG) this quotient group is integrally approximable which we know to be impossible by Example 2.7. This contradiction proves the lemma.  $\square$

**Lemma 5.9.** *Let  $G$  be a totally disconnected locally compact abelian group satisfying  $G = \text{comp}(G)$ . Assume that every compact open subgroup of  $G$  is monothetic. Then  $G$  is inductively monothetic.*

*Proof.* Let  $F$  be a finite subset of  $G$ ; we must show that  $\overline{\langle F \rangle}$  is monothetic. Let  $\mathcal{S}$  be the  $\subseteq$ -directed set of all compact open (and therefore monothetic) subgroups of  $G$ . Since  $G = \bigcup \mathcal{S}$  (see Remark 5.6) for each  $x \in F$  there is a  $C_x \in \mathcal{S}$  such that  $x \in C_x$ . Since  $\mathcal{S}$  is directed and  $F$  is finite there is an  $K \in \mathcal{S}$  such that  $\bigcup_{x \in F} C_x \subseteq K$ . Then  $F \subseteq K$  and so  $\overline{\langle F \rangle} \subseteq K$ . Since  $G$  is totally disconnected, the same is true for  $K$ . By hypothesis,  $K$  is monothetic, and so the comment following Proposition 4.3 shows that  $K$  is inductively monothetic, whence  $\overline{\langle F \rangle}$  is monothetic.  $\square$

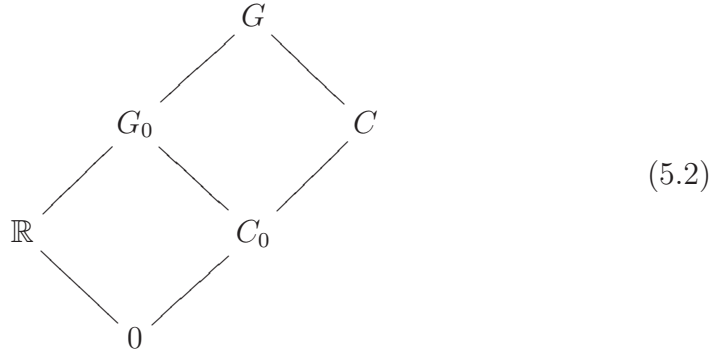
**Corollary 5.10.** *Let  $G$  be an integrally approximable group such that  $\text{comp}(G)$  is periodic. Then  $G/G_0 \cong \text{comp}(G)$  is inductively monothetic.*

*Proof.* Let  $U$  be a compact-open subgroup of  $\text{comp}(G)$ . Then  $G_0 + U \cong \mathbb{R} \times U$  is an open subgroup of  $G = G_0 + \text{comp}(G) = \mathbb{R} \times \text{comp}(G)$ . Hence it is integrally approximable by Proposition 3.1 (OG). So  $U$  is monothetic by Lemma 5.8. Now Lemma 5.9 shows that  $\text{comp}(G)$  is inductively monothetic.  $\square$

Now we have a necessary condition on a locally compact group to be integrally approximable:

**Theorem 5.11.** *Let  $G$  be a nondiscrete integrally approximable locally compact group. Then*

- (a)  $G \cong \mathbb{R} \times \text{comp}(G)$  and
- (b)  $G/G_0 \cong \text{comp}(G)/\text{comp}(G)_0$  is inductively monothetic.



$$C = \text{comp}(G), \quad G = G_0 + C = \mathbb{R} \oplus C.$$

*Remark 5.12.* The Classification of locally compact inductively monothetic groups (Proposition 4.3) yields that the group  $G/G_0$  is of type (3). Indeed, it is a local product of inductively monothetic  $p$ -Sylow subgroups of type  $\mathbb{Z}(p^n)$ ,  $\mathbb{Z}(p^\infty)$ ,  $\mathbb{Z}_p$ , or  $\mathbb{Q}_p$ .

*Proof.* As is described in Proposition 5.5  $G$  has a compact characteristic subgroup  $N \stackrel{\text{def}}{=} \text{comp}(G_0) = \text{comp}(G)_0$ , the unique largest compact connected subgroup. So by Proposition 3.1 (QG), the quotient  $G/N$  is also integrally approximable, and  $\text{comp}(G/N)$  is totally disconnected and therefore is periodic. As Corollary 5.10 applies to  $G/N$ , its factor  $\text{comp}(G/N)$  is inductively monothetic. However,  $G/N \cong \mathbb{R} \times \text{comp}(G/N) = \mathbb{R} \times \text{comp}(G)/N$  and  $G_0 \cong \mathbb{R} \times N$  we see that  $G/G_0 \cong \text{comp}(G/N)$ . Thus  $G/G_0$  is inductively monothetic.  $\square$

It is noteworthy that there is no limitation on the size of the compact connected abelian group  $\text{comp}(G)_0 = \text{comp}(G_0)$ . The locally compact

abelian group  $\text{comp}(G)$  is an extension of the compact group  $\text{comp}(G)_0$  by an inductively monothetic group.

## 6. SUFFICIENT CONDITIONS

In this section we shall prove the following complement to Theorem 5.11 and thereby complete the proof of the main theorem formulated in the abstract.

**Theorem 6.1.** *Let  $G = \mathbb{R} \times H$  for a locally compact abelian group  $H$  satisfying the following conditions:*

- (a)  $H = \text{comp}(H)$  and
- (b)  $H/H_0 \cong G/G_0$  is inductively monothetic.

*Then  $G$  is integrally approximable.*

**6.1. Various reductions.** We shall achieve the proof by reducing the problem step by step.

Firstly, every inductively monothetic group is the directed union of monothetic subgroups by Proposition 4.3. and so if  $H$  satisfies (b) it is of the form  $H = \bigcup_{i \in I} H_i$  with a directed family of subgroups  $H_i \supseteq H_0$  such that  $H_i/H_0$  is compact monothetic. Then, by Proposition 3.1 (DU),  $\mathbb{R} \times H$  is integrally approximable if all  $\mathbb{R} \times H_i$  are integrally approximable for  $i \in I$ .

Thus from here on, in place of condition (b), we shall assume that  $H$  satisfies

- (c)  $H/H_0$  is monothetic.

After condition (c),  $H$  is compact.

Every locally compact abelian group is a strict projective limit of Lie groups. This applies to  $H$ . Clearly condition (a) holds for all quotient groups. If  $N$  is a compact normal subgroup of  $H$  Then  $(H/N)_0 = H_0N/N$  and thus  $(H/N)/(H/N)_0 \cong H/H_0N$ , whence  $(H/N)/(H/N)_0$  is a quotient group of  $H/H_0$  and is therefore inductively monothetic. Thus if we can show that all for all abelian Lie groups  $H$  satisfying (a) and (c), the groups  $\mathbb{R} \times H$  are integrally approximable, then the Proposition 3.1(PL) will show that  $\mathbb{R} \times H$  is integrally approximable. Therefore we need to prove Theorem 6.1 for a compact Lie group  $H$  satisfying (a) and (c).

**Lemma 6.2.** *Let  $H$  be a compact abelian Lie group satisfying (a) and (c). Then there is are nonnegative integers  $m$  and  $n$*

$$H \cong \mathbb{T}^m \times \mathbb{Z}(n).$$



*In particular,  $H$  is monothetic.*

*Proof.*  $H$  is a compact abelian Lie group such that  $H/H_0$  is cyclic. Then  $H$  has the form asserted (see e.g. [14], Proposition 2,42 on p. 48 or Corollary 7.58 (iii) on p. 356).

We have seen in the proof of Lemma 5.8 that a compact group is monothetic if and only if its character group can be injected into the discrete circle group  $\mathbb{T}_d = \mathbb{R}_d \oplus \mathbb{Q}/\mathbb{Z}$ . Since  $\widehat{H} \cong \mathbb{Z}^m \oplus \mathbb{Z}(n)$ , this condition is satisfied.  $\square$

Thus for a proof of Theorem 6.1 it will suffice to prove

**Lemma 6.3.** *If  $H$  is monothetic, then  $\mathbb{R} \times H$  is integrally approximable.*

We shall use the Bohr compactification of the group  $\mathbb{Z}$  of integers. This group is also called the *universal monothetic group*.

Here we shall identify  $\widehat{b\mathbb{Z}}$  with  $\mathbb{T}_d$  (for  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ) and consider the elements  $\chi$  of  $b\mathbb{Z}$  as characters of  $\mathbb{T}_d$ . There is one distinguished character, namely the identity morphism  $\text{id}_{\mathbb{T}}: \mathbb{T}_d \rightarrow \mathbb{T}$ , and we map  $\mathbb{Z}$  naturally and bijectively onto a dense subgroup of  $b\mathbb{Z}$  via the map

$$\rho: \mathbb{Z} \rightarrow b\mathbb{Z}, \quad m \mapsto m \cdot \text{id}. \quad (6.1)$$

The following Lemma shows that for a proof of Lemma 6.3 it suffices to prove it for  $H = b\mathbb{Z}$ .

**Lemma 6.4.** *The group  $G = \mathbb{R} \times H$  is integrally approximable for any monothetic compact subgroup  $H$  if and only if it is so for  $H = b\mathbb{Z}$ .*

*Proof.* Obviously the latter condition is a necessary one for the former, so we have to show that it is sufficient. Let  $H$  be a monothetic group. Then there is a morphism  $f: \mathbb{Z} \rightarrow H$  with dense image. We have the canonical dense morphism  $\rho: \mathbb{Z} \rightarrow b\mathbb{Z}$ . By the universal property of the Bohr compactification there is a unique morphism  $\pi: b\mathbb{Z} \rightarrow H$  such that  $f = \pi \circ \rho$ . Since  $f$  has a dense image, this holds for  $\pi$ , whence by the compactness of  $b\mathbb{Z}$ , the morphism is a surjective morphism between compact groups and therefore is a quotient morphism with a compact kernel. Therefore there is a quotient morphism with compact kernel  $\mathbb{R} \times b\mathbb{Z} \rightarrow \mathbb{R} \times H$ . Hence by Proposition 3.1 (QG),  $\mathbb{R} \times H$  is integrally approximable if  $\mathbb{R} \times b\mathbb{Z}$  is integrally approximable.  $\square$

Now the following lemma will complete the proof of Theorem 6.1 and thereby conclude the section with a proof of the main result of the article.

**Lemma 6.5** (First Key Lemma). *The group  $\mathbb{R} \times \mathfrak{b}\mathbb{Z}$  is approximable by a sequence of integral subgroups.*

**6.2. Proving the First Key Lemma.** The proof of the First Key Lemma requires some technical preparations in which we use the duality of locally compact abelian groups. In the process we need to consider the character group of  $\mathbb{R} \times \mathfrak{b}\mathbb{Z}$ . Here is a reminder of the determination of the character group of a product:

**Lemma 6.6.** *Let  $A$  and  $B$  be locally compact abelian groups. Then there is an isomorphism  $\phi: \widehat{A} \times \widehat{B} \rightarrow (A \times B)^\wedge$  such that*

$$\phi(\chi_A, \chi_B)(a, b) = \chi_A(a) - \chi_B(b).$$

We apply this with  $A = \mathbb{R}$  and  $B = \mathfrak{b}\mathbb{Z}$ . For the simplicity of notation we shall denote the coset  $r + \mathbb{Z} \in \mathbb{T}$  of  $r \in \mathbb{R}$  by  $\bar{r}$ . We consider  $\mathbb{R}$  also as the character group of  $\mathbb{R}$  by letting  $r(s) = \bar{r}s \in \mathbb{T}$ . We also identify  $\widehat{\mathbb{T}}$  with  $\mathbb{Z}$  by considering  $k \in \mathbb{Z}$  as the character defined by  $k(\bar{r}) = \overline{kr}$ . Recall that we consider  $\mathfrak{b}\mathbb{Z}$  as the character group of  $\mathbb{T}_d$ . In the spirit of Lemma 6.6, we identify  $\mathbb{R} \times \mathbb{T}_d$  with the character group  $\widehat{G}$  of  $G = \mathbb{R} \times \widehat{\mathbb{T}_d} = \mathbb{R} \times \mathfrak{b}\mathbb{Z}$  by letting  $(r, \bar{s})$  denote the character of  $G$  defined by  $(r, \bar{s})(x, \chi) = \bar{r}x - \chi(\bar{s}) \in \mathbb{R}/\mathbb{Z}$ . We recall that the identity function  $\text{id}_{\mathbb{T}}: \mathbb{T}_d \rightarrow \mathbb{T}$  is a particular character of  $\mathbb{T}_d$  and thus is an element of  $\mathfrak{b}\mathbb{Z}$ ; indeed  $\text{id}_{\mathbb{T}}$  is the distinguished generator of  $\mathfrak{b}\mathbb{Z}$ . Now we define

$$Z_n = \mathbb{Z} \cdot \left(\frac{1}{n}, \text{id}_{\mathbb{T}}\right) \in \mathcal{F}_1(G), \quad G = \mathbb{R} \times \mathfrak{b}\mathbb{Z}. \quad (6.2)$$

Accordingly,  $(r, \bar{s}) \in \mathbb{R} \times \mathbb{T}_d$  belongs to the annihilator  $Z_n^\perp = (\frac{1}{n}, \text{id}_{\mathbb{T}})^\perp$  iff  $\frac{\bar{r}}{n} - \bar{s} = 0$ , that is, iff  $\frac{r}{n} = s$ . This means  $\frac{r}{n} + \mathbb{Z} = s + \mathbb{Z}$ , and so

$$(r, \bar{s}) \in \mathbb{R} \times \mathbb{T}_d \quad \text{is in } Z_n^\perp \quad \text{iff} \quad \frac{r}{n} - s \in \mathbb{Z}. \quad (6.3)$$

In an effort to show that  $\lim_n Z_n^\perp = \{0\}$  in  $\mathcal{SUB}(\widehat{G})$  we consider the following

**Lemma 6.7** (Convergence to the trivial subgroup). *Let  $\Gamma$  be a locally compact group with identity  $e$ , and let  $(H_n)_{n \in \mathbb{N}}$  be a sequence of closed subgroups. Then the following statements are equivalent:*

- (a) *For each subnet  $(H_{n_j})_{j \in J}$  of  $(H_n)_{n \in \mathbb{N}}$  and each convergent net  $(h_{n_j})_{j \in J}$  with  $h_{n_j} \in H_{n_j}$  and limit  $h$  we have  $h = e$ .*
- (b)  *$\lim_{n \in \mathbb{N}} H_n = \{e\}$  in  $\mathcal{SUB}(G)$ .*

*Proof.* (a)  $\Rightarrow$  (b): We argue by contradiction. Suppose that there is a compact subset  $K$  of  $G$  and an open neighborhood  $U$  of the identity such that for each  $\alpha \in \mathbb{N}$  there is an  $n_\alpha \in \mathbb{N}$  with  $\alpha \leq n_\alpha$  such that  $H_{n_\alpha} \notin \mathcal{U}(\{e\}; K, U)$ . That is,  $H_{n_\alpha} \cap K \not\subseteq U$  by equation (2.3), and so there is an  $h_{n_\alpha} \in H_{n_\alpha}$  such that  $h_{n_\alpha} \in K \setminus U$ . As  $K \setminus U$  is a compact subset of  $G$ , the net  $(h_{n_\alpha})$  admits a subnet converging to a point  $a$  of  $K \setminus U$ . Since  $e \notin K \setminus U$ ,  $a \neq e$ , which is a contradiction.

(b)  $\Rightarrow$  (a): Let  $(h_{n_j})_{j \in J}$  be a net converging to  $h$  and assume  $h_{n_j} \in H_{n_j}$  for every  $j \in J$ . Suppose  $h \neq e$ . Now let  $K$  be a compact neighborhood of  $h$  not containing  $e$ . We may assume that  $h_{n_j} \in K$  for all  $j \in J$ . As  $\lim_{n \in \mathbb{N}} H_n = \{e\}$ , there exists  $N \in \mathbb{N}$  such that for each  $n \geq N$  we have  $H_n \in \mathcal{U}(\{e\}; K, \Gamma \setminus K)$ , that is,  $H_n \cap K \subseteq \Gamma \setminus K$  by equation (2.3), and this is a contradiction.  $\square$

In order to appreciate this lemma, consider the condition

(a') For each convergent net  $(h_i)_{i \in I}$  with  $h_i \in H_i$  and limit  $h$  we have  $h = e$ .

The following example will show that the implication (a')  $\Rightarrow$  (b) fails.

**Example 6.8.** In  $\mathcal{SUB}(\mathbb{R})$ , let

$$H_n = \begin{cases} n\mathbb{Z}, & \text{if } n \text{ is even;} \\ \frac{1}{n}\mathbb{Z}, & \text{if } n \text{ is odd.} \end{cases}$$

Let  $(h_n)$  be a sequence in  $\mathbb{R}$  converging to  $h$  and such that, for each  $n$ ,  $h_n \in H_n$ . For any  $n \in \mathbb{N}$ , there is  $k_n \in \mathbb{Z}$  such that  $h_{2n} = 2nk_n$ . As the subsequence  $(h_{2n})$  converges to  $h$ ,  $h = 0$ . So (a') is satisfied. However, as the subsequence  $(H_{2n})$  converges to  $\{0\}$  and the subsequence  $(H_{2n+1})$  converges to  $\mathbb{R}$ , the sequence  $(H_n)$  is divergent and so (b) fails.

**Lemma 6.9.**  $\lim_{n \in \mathbb{N}} Z_n^\perp = \{0\}$ .

*Proof.* We shall apply Lemma 6.7 and assume that we have a net  $(n_j)_{j \in J}$  cofinal in  $\mathbb{N}$  such that  $(r_j, \overline{s}_j)_{j \in J}$  is a convergent net with

$$(r_j, \overline{s}_j) \in Z_{n_j}^\perp \text{ for all } j \in J \text{ and with } (r, \overline{s}) = \lim_{j \in J} (r_j, \overline{s}_j).$$

From Equation (6.3) we know that  $\frac{r_j}{n_j} - s_j \in \mathbb{Z}$ . Since  $r_j \rightarrow r$ ,  $s_j \rightarrow s$ , and  $n_j \rightarrow \infty$ , we conclude  $s \in \mathbb{Z}$  and thus  $\overline{s} = 0$ . Further  $s \in \mathbb{Z}$  and  $r_j/n_j \rightarrow 0$  imply the existence of a  $j_0$  such that  $j_0 \leq j$  implies  $s_j = s$ . Then  $r_j/n_j$  is an integer, and so for large enough  $j$  we have  $r_j = 0$  which implies  $r = 0$ . Thus  $\lim_{j \in J} (r_j, \overline{s}_j) = 0$  and by Lemma 6.7 this shows that  $\lim_{n \in \mathbb{N}} Z_n^\perp = \{0\}$  which we had to show.  $\square$

Now we are ready for proof of the First Key Lemma: There is a sequence  $(Z_n)_{n \in \mathbb{N}}$  in  $\mathcal{F}_1(G)$ ,  $Z_n = \mathbb{Z} \cdot (\frac{1}{n}, \text{id}_T)$  for which  $(Z_n^\perp)_{n \in \mathbb{N}}$  converges

to  $\{0\}$  in  $\mathcal{SUB}(\widehat{G})$  by Lemma 6.9. Now we apply Pontryagin-Chabauty Duality in the form of Proposition 5.3 and conclude

$$\lim_{n \in \mathbb{N}} Z_n = G \text{ in } \mathcal{SUB}(G) \text{ for } G = \mathbb{R} \times b\mathbb{Z}. \quad (6.4)$$

This concludes the proof of the First Key Lemma 6.5 and thus also finishes the proof of Theorem 6.1.

We can summarize the main result as follows:

**Main Theorem A.** *A locally compact group  $G$  is integrally approximable if and only if it is either discrete, in which case it is isomorphic to a nonsingleton subgroup of  $\mathbb{Q}$ , or else it is abelian and of the form  $G \cong \mathbb{R} \times \text{comp}(G)$  where  $G/G_0 \cong \text{comp}(G)/\text{comp}(G)_0$  is periodic inductively monothetic.*

We recall that the groups  $\text{comp}(G)_0$  range through all compact connected abelian groups and that the periodic inductively monothetic groups were classified in Proposition 4.3 (3).

We mention in passing that Theorem A yields a characterisation of the the group  $\mathbb{R}$  in the class of locally compact groups. For this purpose let us call a topological group *compact-free* if it does not contain a nonsingleton compact subgroup.

**Corollary 6.10.** *For a locally compact group  $G$  the following conditions are equivalent:*

- (1)  $G \cong \mathbb{R}$  or else is isomorphic to a nonsingleton subgroup of the discrete group  $\mathbb{Q}$ .
- (2)  $G$  is compact-free and integrally approximable.

A similar characterization using the property of being compact-free was suggested by Chu in [4].

## 7. COMPLEMENT: GROUPS APPROXIMABLE BY REAL SUBGROUPS

We classified locally compact groups  $G$  for which every neighborhood of  $G \in \mathcal{SUB}(G)$  contains a subgroup  $H$  of  $G$  isomorphic to  $\mathbb{Z}$ . We called such groups *integrally approximable*. Now we shall do the same for groups  $G$  with the same property except that  $\mathbb{Z}$  is replaced by  $\mathbb{R}$ . We need a name for these groups that are approximated by subgroups of (real) *numbers*. For this purpose let us denote by  $\mathcal{R}_1(G)$  the subspace of  $\mathcal{SUB}(G)$  containing all subgroups isomorphic to the group  $\mathbb{R}$  of all real numbers, the *real vector group of dimension 1*.

**Definition 7.1.** A locally compact group  $G$  is said to be *numerally approximable* if  $G \in \overline{\mathcal{R}_1(G)}$ .

Trivially,  $\mathbb{R}$  is numerally approximable. The theory and classification of numerally approximable groups is in most ways simpler than that of integrally approximable groups. The basic aspects are completely analogous to the former and that allows us now to proceed more expeditiously.

**Example 7.2.** For  $n \in \mathbb{N}$  and  $G = \mathbb{R}^n$ , the following statements are equivalent:

- (1)  $G$  is numerally approximable.
- (2)  $n = 1$ .

*Proof.* Trivially, (2) implies (1). Conversely, if (1) is satisfied, an inspection of the proof of (1) $\Rightarrow$ (2) for Example 2.5 shows that it applies to the next to the last sentence, where  $\mathbb{Z}$  needs to be replaced by  $\mathbb{R}$  to apply literally.  $\square$

**Lemma 7.3.** *Let  $G$  be a locally compact group and  $(A_i)_{i \in I}, (B_i)_{i \in I}$  two nets converging to  $A$  and  $B$  respectively. If  $A_i \subseteq B_i$  holds eventually, then  $A \subseteq B$ .*

*Proof.* By way of contradiction, suppose that  $A$  is not a subgroup of  $B$ . Let  $x \in A \setminus B$  and  $U$  a relatively compact open neighborhood of  $e$  such that  $\overline{U}x \cap B = \emptyset$ . As  $B_i \rightarrow B$ , there exists  $i_0 \in I$  such that for any  $i \geq i_0$  we have  $\overline{U}x \cap B_i = \emptyset$  and so  $Ux \cap A_i = \emptyset$ , which is a contradiction because the set of all closed subgroups meeting the open set  $Ux$  is an open neighborhood of  $A$ .  $\square$

In particular, this lemma implies that the limit of a net of connected closed subgroups of a locally compact group  $G$  is contained in the identity component  $G_0$  of  $G$ , whence a group  $G$  which is approximable by real subgroups is necessarily connected. Using also Proposition 5.1, we conclude:

**Lemma 7.4.** *Every numerally approximable locally compact group is abelian and connected.*

In view of Proposition 5.4 we may rephrase this as follows:

**Theorem 7.5.** *Let  $G$  be a numerally approximable locally compact group. Then  $\text{comp}(G)$  is compact and connected and*

$$G \cong \mathbb{R} \times \text{comp}(G).$$

$$\begin{array}{ccc}
 & G & \\
 \mathbb{R} & & C \\
 & 0 &
 \end{array}
 \tag{7.1}$$

$$C = \text{comp}(G), \quad G = \mathbb{R} \oplus C.$$

The second part of the structure theorem for numerally approximable groups, saying that every group  $\mathbb{R} \times C$  with any compact connected abelian group  $C$  is numerally approximable is proved in a reduction procedure similar to the one we used for integrally approximable groups.

**Theorem 7.6.** *Let  $G = \mathbb{R} \times C$  for a compact connected abelian group  $C$ . Then  $G$  is numerally approximable.*

*Proof.* Every compact connected group is a directed union of compact connected monothetic groups (see e.g. [13], Theorem I or [14], Theorem 9.36(ix), pp. 479f.). Thus  $G$  is the directed union of subgroups  $\mathbb{R} \times M$  where  $M$  is compact connected monothetic. The closure lemma Proposition 3.1 (DU) is easily seen to apply to numerally approximable groups in place of integrally approximable groups. Therefore it is no loss of generality to assume that  $C$  is compact connected monothetic. Then it is a quotient of  $\text{b}\mathbb{R}$ , the Bohr compactification of  $\mathbb{R}$ . (Indeed  $\widehat{C}$  is a discrete torsion free group of rank  $\leq 2^{\aleph_0}$  and thus is a subgroup of  $\mathbb{R}_d$  (the discrete reals), and thus  $C$  is a quotient of  $\widehat{\mathbb{R}_d} \cong \text{b}\mathbb{R}$ .) Since the closure lemma Proposition 3.1 (QG) again applies to numerally approximable groups in place of integrally approximable groups, the proof will be complete if it is shown that  $\mathbb{R} \times \text{b}\mathbb{R}$  is numerally approximable. This will be done in the Second Key Lemma that follows.  $\square$

The proof of the Theorem is therefore reduced to showing that one special group in numerally approximated:

**Lemma 7.7** (Second Key Lemma). *The group  $G = \mathbb{R} \times \text{b}\mathbb{R}$  is numerally approximable.*

Before we prove this Second Key Lemma, we review the duality aspects of the present situation.

From Lemma 6.6 we recall that the dual  $\widehat{G}$  of  $G$  may be identified with  $\mathbb{R} \times \mathbb{R}_d$  where, as before we identify  $\widehat{\mathbb{R}}$  and  $\mathbb{R}$ . Let  $f: \mathbb{R} \rightarrow \text{b}\mathbb{R}$  the

canonical one-parameter subgroup of  $\mathfrak{b}\mathbb{R}$ , namely the dual of  $\text{id}_{\mathbb{R}}: \mathbb{R}_d \rightarrow \mathbb{R}$ . For each natural number  $n \in \mathbb{N}$ , the morphism  $n \cdot f$  defined by  $(n \cdot f)(r) = n \cdot f(r)$  (in the additively written) abelian group  $\mathfrak{b}\mathbb{R}$ ) is the dual of the morphism  $n \cdot \text{id}_{\mathbb{R}}: \mathbb{R}_d \rightarrow \mathbb{R}$  which is just multiplication by  $n$ . We let the subgroup  $R_n \leq \mathbb{R} \times \mathfrak{b}\mathbb{R}$  be the graph of  $n \cdot f$ , that is,

$$R_n = \{(r, n \cdot f(r)) \mid r \in \mathbb{R}\}.$$

Clearly,  $R_n$  is isomorphic to  $\mathbb{R}$ , since the projection of a graph of a morphism onto its domain is always an isomorphism.

We claim that  $G = \lim_n R_n$  in  $\mathcal{SUB}(G)$ ; this claim will finish the proof of the Second Key Lemma. We shall prove the claim by showing that  $\lim_n R_n^\perp = \{(0, 0)\}$  in  $\mathcal{SUB}(\widehat{G})$ . This will prove the claim by Pontryagin-Chabauty-Duality in the form of Proposition 5.3.

We need information on the annihilator of a graph:

**Lemma 7.8.** *Let  $A$  and  $B$  be locally compact groups and  $f: A \rightarrow B$  a morphism. Define  $\Gamma = \{(a, f(a)) \mid a \in A\} \subseteq A \times B$  to denote the graph of  $f$ . We identify  $(A \times B)^\wedge$  with  $\widehat{A} \times \widehat{B}$  (via  $\rho$  as in Lemma 6.6). Then the graph  $\{(\widehat{f}(b), b) \mid b \in B\} \subseteq A \times B$  of the adjoint morphism  $\widehat{f}: \widehat{B} \rightarrow \widehat{A}$  is the annihilator  $\Gamma^\perp$  of  $\Gamma$ .*

*Proof.* An element  $(\chi_A, \chi_B) \in (A \times B)^\times$  is in  $\Gamma^\perp$  if and only if, for any  $a \in A$ ,

$$0 = (\chi_A, \chi_B)(a, f(a)) = \chi_A(a) - (\chi_B \circ f)(a) = (\chi_A - \widehat{f}(\chi_B))(a),$$

that is  $\chi_A = \widehat{f}(\chi_B)$ . □

Applying this lemma to the graph  $R_n$  we see that

$$R_n^\perp = \{(nr, r) : r \in \mathbb{R}\} = \{(r, \frac{r}{n}) : r \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}_d. \quad (7.2)$$

Note that  $R_n^\perp = \text{graph}(r \mapsto \frac{r}{n})$ .

**Lemma 7.9.** *We have  $\lim_n R_n^\perp = \{(0, 0)\}$  in  $\mathcal{SUB}(\widehat{G})$ .*

*Proof.* As in the proof of Lemma 6.9 we invoke Lemma 6.7 and consider a net  $(n_j)_{j \in J}$  cofinal in  $\mathbb{N}$  such that  $(r_j, s_j)_{j \in J}$  is a convergent net in  $\widehat{G} = \mathbb{R} \times \mathbb{R}_d$  such that according to Equation (7.2) we have

- (a)  $(r_j, s_j) \in R_{n_j}^\perp = \{(r, \frac{r}{n_j}) : r \in \mathbb{R}\}$  for all  $j \in J$ , and
- (b)  $(r, s) = \lim_{j \in J} (r_j, s_j)$  in  $\widehat{G} = \mathbb{R} \times \mathbb{R}_d$ .

We must show that this implies  $r = s = 0$ ; then Lemma 6.7 completes the proof of the lemma.



Now (a) implies  $s_j = r_j/n_j$  for all  $j \in J$ , and (b) yields, firstly, that  $r = \lim_j r_j$  in  $\mathbb{R}$  and, secondly, that in view of the discreteness of  $\mathbb{R}_d$  the net of the  $r_j/n_j = s_j$  in  $\mathbb{R}_d$  are eventually constant, say  $= t$  for  $j > j_0$  for some  $j_0$ . For these  $j$  we now have  $r_j = tn_j$ , and so  $r = \lim_j tn_j$ . Since the  $n_j$  increase beyond all bounds, this implies  $r = t = 0$ , and so  $r_j = 0$  for  $j > j_0$ . Accordingly,  $s_j = 0$  for  $j > j_0$ , and thus  $s = \lim_j s_j = 0$  as well. This completes the proof.  $\square$

By our earlier remarks, this shows  $\lim_n R_n = G$  and thus completes the proof of the Second Key Lemma 7.7 and thereby also completes the proof of Theorem 7.6.

We can summarize the material on numerally approximable groups as follows:

**Main Theorem B.** *A locally compact group  $G$  is numerally approximable if and only if it is of the form  $G \cong \mathbb{R} \times C$  with a compact connected abelian group  $C$ .*

By Pontryagin Duality, the groups  $C$  range through a class equivalent to the class of all torsion free abelian (discrete) groups.

A comparison of Main Theorems A and B allow us to draw the following conclusion:

**Corollary 7.10.** *If a locally compact group is numerally approximable, then it is integrally approximable, while the reverse is not generally true.*

*Remark 7.11.* Locally compact groups of the form  $\mathbb{R} \times C$  for a compact connected group  $C$  have been called *two-ended* by Freudenthal ([6]).

## 8. AN ALTERNATE PROOF OF COROLLARY 7.10

Corollary 6.10 was proved above in a roundabout fashion. We therefore present a different direct way to arrive at the same conclusion.

**8.1. Iterated Limit Theorem.** The Iterated Limit Theorem for nets deals with the following data:

Let  $I$  be a directed set and  $(J_i)_{i \in I}$  a family of directed sets indexed by  $I$ . Assume that for each  $i \in I$  we are given a converging net  $(x_{ij})_{j \in J_i}$  in a topological space  $X$ . The limits  $r_i = \lim_{j \in J_i} x_{ij}$ ,  $i \in I$  form a net  $(r_i)_{i \in I}$  which may or may not converge. If it does, we have what is sometimes called an iterated limit

$$r = \lim_{i \in I} \lim_{j \in J_i} x_{ij}.$$

There is no harm in our assuming that the sets  $J_i$  are pairwise disjoint. (If necessary we can replace each  $J_i$  by  $\{i\} \times J_i$ !). Now  $D \stackrel{\text{def}}{=} \dot{\bigcup}_{k \in I} J_k$  is a disjoint union of the fibers  $J_i$  of the fibration  $\pi: D \rightarrow I$ ,  $\pi(d) = i$  iff  $d \in J_i$ . A function  $\sigma: I \rightarrow D$  is a *section* if  $\pi \circ \sigma = \text{id}_I$ , the identity function of  $I$ . The set of sections is  $\prod_{i \in I} J_i$ . The set  $D$  is partially ordered lexicographically:

(1)  $d \leq d'$  if either  $\pi(d) < \pi(i')$ , or  $\pi(i) = \pi(i')$  and  $d \leq d'$  in  $J_{\pi(d)}$ .

These data taken together result in a net  $(x_d)_{d \in D}$  which converges fiberwise, and the limits along the fibers form a convergent net.

In applications such as ours it is desirable to construct from the given data a subnet  $(y_p)_{p \in P}$  of the net  $(x_d)_{d \in D}$  in  $X$  for some directed set  $P$  such that

$$r = \lim_{p \in P} y_p.$$

**Lemma 8.1.** *Let  $P = I \times \prod_{k \in I} J_k$ . For  $p = (i, \sigma) \in P$  we define  $d_p = \sigma(i) \in D$ . Then  $p \mapsto d_p: P \rightarrow D$  is a cofinal function between directed sets, that is, for each  $d \in D$  there is a  $p_0 \in P$  such that  $p_0 \leq p$  implies  $d \leq d_p$ .*

*Proof.* Let  $d \in D$ , that is,  $d \in J_{\pi(d)}$ . We have to find a  $p_0 = (i, \sigma) \in P$  such that  $p_0 \leq p = (k, \tau)$  in  $P$  implies  $\sigma(i) \leq \tau(k)$  in  $D$ .

Now for  $k \neq i$  use the Axiom of Choice to select an arbitrary  $j_k \in J_k$ . Define  $\sigma \in \prod_{i' \in I} J_{i'}$  by

$$\sigma(k) = \begin{cases} \sigma(\pi(d)), & \text{if } k = \pi(i) \\ j_k, & \text{otherwise.} \end{cases}$$

Set  $p_0 = (\pi(d), \sigma)$ . Now if  $p_0 \leq p = (k, \tau)$  with  $\pi(d) \leq k$  and  $\sigma \leq \tau$  we claim that  $d = \sigma(\pi(d)) \leq d_p = \tau(k)$ . Indeed, if  $\pi(d) < k$  then  $d \leq \tau(k) = d_p$  by the order on  $D$ , and if  $i = k$  then  $d = \sigma(\pi(d)) \leq \tau(\pi(d))$  since  $\sigma \leq \tau$ . This completes the proof.  $\square$

The subnet is constructed so as to be convergent if the iterated limit exists and to have the same limit. This is the so-called *Iterated Limit Theorem* in whose formulation we use the notation introduced above.

**Theorem 8.2.** *Assume that the net  $(x_d)_{d \in D}$  converges fiberwise and that the fiberwise limits converge as well. Then it has a subnet  $(y_p)_{p \in P}$ ,  $y_p = x_{d_p}$ ,  $d_{i, \sigma} = \sigma(i)$ , such that*

$$\lim_{p \in P} y_p = \lim_{i \in I} \lim_{d \in J_i} x_d.$$

*Proof.* For a proof see [15], p. 69.  $\square$

In this remarkable fact about nets and their convergence it is noteworthy that the index set  $P$  is vastly larger than the already large index set  $D$ .

A frequent special case is that the index sets  $J_i$  all agree with one and the same index set  $J$  in which case we have  $D = I \times J$ ,  $J_i = \{i\} \times J$ , and  $P = I \times J^I$ .

Now we present an alternate proof of Corollary 7.10.

*Proof.* Let  $G$  be a numerally approximable locally compact group and let  $(R_j)_{j \in J}$  be a net such that  $G = \lim_{j \in J} R_j$  and  $R_j \cong \mathbb{R}$ . For each  $j \in J$  we have  $R_j = \lim_{n \in \mathbb{N}} Z_{(j,n)}$  for a sequence  $Z_{(j,n)} \cong \mathbb{Z}$ . Therefore

$$G = \lim_{j \in J} \lim_{n \in \mathbb{N}} Z_{(j,n)}.$$

Then there exists a subnet  $(Z_p)_{p \in P}$  of the net  $(Z_{(j,n)})_{(j,n) \in J \times \mathbb{N}}$  such that

$$G = \lim_{p \in P} Z_p.$$

by the Theorem of the Iterated Limit.  $\square$

A review of the iterated limit theorem together with an application of it in the context of our present topic may be of independent interest.

## REFERENCES

1. BRACONNIER, J, *Sur les groupes topologiques localement compacts*, J. Math. Pures Appl. 9) **27** (1948), 1–85.
2. BRACONNIER, J, *Sur les groupes topologiques localement compacts*, Thèses françaises de l'entre-deux-guerres, 1945, iv p+1–121.
3. BRIDSON M., DE LA HARPE P., KLEPTSYN V., *The Chabauty space of closed subgroups of the three-dimensional Heisenberg group*, Pacific J. Math., **240**(2009), 1–48.
4. CHU, H. *A characterisation of integer groups and real groups* Duke Math. J. **28** (1961), 125–132.
5. CORNULIER Y., *On the Chabauty space of locally compact abelian groups*, Algebraic and Geometric Topology, **11** (2011), 2007–2035.
6. FREUDENTHAL, H. *La structure des groupes á deux bouts et des groupes triplement transitifs*, Indag. Math **13** (1951), 288–294.
7. HAMROUNI H., KADRI B., *On the compact space of closed subgroups of locally compact groups*, J. Lie Theory, **24** (2014), 715–723.
8. HAMROUNI H., *On a question of Ross and Stromberg*, J. Lie Theory, **25** (2015), 889–901.
9. HAMROUNI H., KADRI B., *Discretely approximable locally compact groups and Jessen's theorem for nilmanifolds*, Bull. Sci. Math., **140** (2016), 1–13.
10. HAMROUNI H., SADKI F., *On the continuity of the centralizer map of a locally compact group*, J. Lie Theory, **26** (2016), 117–134.

11. HATTEL T., *L'espace des sous-groupes fermés de  $\mathbb{R} \times \mathbb{Z}$* , Algebr. Geom. Topol. **10** (2010), 1395–1415.
12. HERFORT W., HOFMANN, K. H., AND RUSSO, F. G. *Locally Compact Near Abelian Groups*, in preparation (2016).
13. HOFMANN K. H., *Connected abelian groups in compact loops*, Trans. Amer. Math. Soc, **104** (1962), 132–143.
14. HOFMANN K. H., MORRIS S. A., *The structure of compact groups*, De Gruyter, Berlin, 1998, Third Edition, Revised and Augmented, 2013.
15. KELLEY J. L., *General topology*, Graduate Texts in Mathematics (Book 27), Springer (1975).
16. MONTGOMERY D., ZIPPIN L., *Topological transformation groups*, Interscience Tracts in Pure and Applied Mathematics 1, Interscience Publishers, New York 1955.

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